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On the Connectedness of Square Element Graphs over Arbitrary Rings

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Abstract. In this paper, we continue our study of the square element graph which is defined as follows: if R is a ring then the square element graph $\mathbb{S}q(R)$ is the simple undirected graph whose vertex set consists of all non-zero elements of R, and two vertices u, v are adjacent if and only if $u + v = x^2$ for some $x \in R - \{0\}$. Here we mainly consider the connectedness of $\mathbb{S}q(R)$ over different rings. We particularly look at $\mathbb{S}q(R)$ taken over some infinite rings which contain $\mathbb{Z}$ as a subring. For a ring R and an ideal I of R, the relation between the connectedness of $\mathbb{S}q(R)$ and $\mathbb{S}q(R/I)$ is studied. Finally, connectedness of $\mathbb{S}q(R)$ over various polynomial rings and matrix rings are considered.

Keywords: Square element graph; Infinite graph; Connectedness; Diameter.

1. Introduction

In the last few decades, several graphs have been defined over algebraic structures (for example, see [1]–[14]). Among these, the square element graph seems to be the first instance where the set of square elements of a ring is directly associated with a graph. Sen Gupta and Sen [11] first introduced the square element graph over a finite commutative ring and studied the interplay between its graph-theoretic properties and the algebraic properties. After that, in [12], they generalized the square element graph by defining it over any ring. The square element graph is defined in the following way:

Definition 1.1. [12] *Let R be a ring. The Square element graph over R is the simple undirected graph $G = (V, E)$, where $V = R - \{0\}$ and $ab \in E$ if and only if $a \neq b$ and $a + b = x^2$ for some $x \in R - \{0\}$. We denote this graph by $\mathbb{S}q(R)$.*

Sen Gupta and Sen studied several properties of $\mathbb{S}q(R)$ in [12]. In particular, they considered $\mathbb{S}q(R)$ defined over some important infinite rings. In this paper, we continue the study of $\mathbb{S}q(R)$. We give some significant results regarding the connectedness of $\mathbb{S}q(R)$ for some infinite rings (in particular, infinite fields, polynomial rings and matrix rings), and in this process we provide some shorter and alternative proofs of some results given in the paper [12]. We also consider the relation between the connectedness of the square element graph taken over any ring R and that taken over any of its quotient rings.

Throughout this paper, $Char(R)$ denotes the characteristic of a ring R. Also $a \leftrightarrow b$ denotes that the vertices a and b are adjacent. $N(v)$ denotes the set of vertices adjacent with the vertex v. For other graph-theoretic terminologies, we refer to [14].

2. Connectedness of $\mathbb{S}q(R)$ over Infinite Rings

In this section, we consider the connectedness of the graph $\mathbb{S}q(R)$ defined over different infinite rings R.

In [12], Sen Gupta and Sen proved that $\mathbb{S}q(\mathbb{Z})$ is connected. Here we give an alternative proof of the same using the well-known Lagrange's four-square theorem.

Theorem 2.1. $\mathbb{S}q(\mathbb{Z})$ *is connected.*

Proof. We show that for any vertex v $(\neq -1)$, there is a path between v and -1 in $\mathbb{S}q(\mathbb{Z})$. Since v is an integer, by Lagrange's four-square theorem it follows that $|v|$ is a sum of four integer squares. First, let $v > 0$ (note that $v \neq 0$). Then $v = x^2 + y^2 + u^2 + t^2$ for some integers x, y, u, t. Since $v \neq 0$, at least one of x, y, u, t is non-zero. Suppose v is a square integer itself. Then we can take $y = u = t = 0$ and $v = x^2$. In that case we have a walk $v \leftrightarrow 8x^2 \leftrightarrow -4x^2 \leftrightarrow 5x^2 \leftrightarrow 4x^2 \leftrightarrow -4x+1 \leftrightarrow 4x \leftrightarrow 1+4x^2 \leftrightarrow -1$ between v and -1. If v is not a square integer, then it follows that at least two of x, y, u, t are non-zero. Without loss of generality, let $x, y \neq 0$. If $v = x^2 + y^2$, then we have a walk $v \leftrightarrow 3x^2 - y^2 \leftrightarrow 6x^2 + y^2 \leftrightarrow -6x^2 \leftrightarrow 9x^2 \leftrightarrow 7x^2 \leftrightarrow -3x^2 \leftrightarrow 4x^2 \leftrightarrow -4x+1 \leftrightarrow 4x \leftrightarrow 1+4x^2 \leftrightarrow -1$. Again, if $x, y, t \neq 0$, then we have a walk $v \leftrightarrow -x^2 - u^2 - t^2 \leftrightarrow 2x^2 + u^2 + t^2 \leftrightarrow -2x^2 - u^2 \leftrightarrow 3x^2 + u^2 \leftrightarrow -3x^2 \leftrightarrow 4x^2 \leftrightarrow -4x+1 \leftrightarrow 4x \leftrightarrow 1+4x^2 \leftrightarrow -1$. So for any $v > 0$, there exists a walk (and hence, a path) between v and -1 in $\mathbb{S}q(\mathbb{Z})$. Again, let $v < -1$. Then $1 - v$ is a positive integer. As we have just shown, there exists a path (say, P) between $1 - v$ and -1 in $\mathbb{S}q(\mathbb{Z})$. Thus we have a walk $v \leftrightarrow 1 - v \leftrightarrow P \leftrightarrow -1$ between v and -1. This shows that $\mathbb{S}q(\mathbb{Z})$ is connected. ∎

Next, we consider the connectedness of $\mathbb{S}q(F)$ for a field of characteristic 0.

Theorem 2.2. *Let F be a field with $Char(F) = 0$. Then $\mathbb{S}q(F)$ is connected with $diam(\mathbb{S}q(F)) \leq 7$.*

Proof. We show that for any $a \in F - \{0, -1\}$, there exists a path between a and -1 in $\mathbb{S}q(F)$. Let $a \in F - \{0, -2\}$ and $4^{-1}a^2 \neq -1$. Then we have a path $a \leftrightarrow 4^{-1}a^2 + 1 \leftrightarrow -1$. Thus there exists a path of length at most 2 between a and -1 for any $a \in F - \{0, -2\}$ with $4^{-1}a^2 \neq -1$. Now considering the vertex -2, we have a path $-2 \leftrightarrow 3 \leftrightarrow 4^{-1}(3^2) + 1 \leftrightarrow -1$. Finally, if $4^{-1}a^2 = -1$, then we have a path $a \leftrightarrow a^2 + 4^{-1} \leftrightarrow -4^{-1} \leftrightarrow 4^{-1}((4^{-1})^2) + 1 \leftrightarrow -1$. Hence all the vertices are in the component to which the vertex -1 belongs. This shows that $\mathbb{S}q(F)$ is connected. Now let a, b be any two vertices distinct from -1. If $a = -b$ and $a^2 = -4$, then we have a path $b \leftrightarrow a^2 + 4^{-1} \leftrightarrow a$. Let $a \neq -b$ and (without loss of generality) $a^2 \neq -4$. Then from the paths considered before, we see that there is a path of length at most 4 from a to -1 and a path of length at most 3 from b to -1. Thus we have a path of length at most 7 between any two vertices of $\mathbb{S}q(F)$. Hence $diam(\mathbb{S}q(F)) \leq 7$. ∎

Corollary 2.3. $\mathbb{S}q(\mathbb{Q})$ *is connected and* $diam(\mathbb{S}q(\mathbb{Q})) \leq 7$.

Remark 2.4. Note that in [12], Sen Gupta and Sen had used the connectedness of $\mathbb{S}q(\mathbb{Z})$ to prove the connectedness of $\mathbb{S}q(\mathbb{Q})$. But Theorem 2.2 gives us an independent (and shorter) proof of connectedness of $\mathbb{S}q(\mathbb{Q})$. Moreover, it gives us a bound for the diameter of $\mathbb{S}q(\mathbb{Q})$.

By Theorem 2.2, it follows that $\mathbb{S}q(\mathbb{R})$ is connected. In fact, we can show that $diam(\mathbb{S}q(\mathbb{R})) = 2$.

Theorem 2.5. $\mathbb{S}q(\mathbb{R})$ *is connected with diameter 2.*

Proof. It is known that for any $x \in \mathbb{R}^+$, there exists some $y \in \mathbb{R}^+$ such that

$x = y^2$. Consider two vertices u, v in $\mathbb{S}q(\mathbb{R})$. If $u, v > 0$, then $u \leftrightarrow v$. If $u > 0$ and $v < 0$ (without loss of generality), then we have a path $u \leftrightarrow |v| + 1 \leftrightarrow v$. Finally, if $u, v < 0$, then we have a path $u \leftrightarrow |uv| + 1 \leftrightarrow v$. Hence we have a path of length at most 2 between any two vertices of $\mathbb{S}q(\mathbb{R})$. Also, $\mathbb{S}q(\mathbb{R})$ is not complete since $-2 \not\leftrightarrow 1$. Hence $\mathbb{S}q(\mathbb{R})$ is connected with diameter 2. ∎

The field $\mathbb{C}$ is algebraically closed and hence every non-zero element of $\mathbb{C}$ is a square element of $\mathbb{C}$. Noting that 0 is not a square element, this leads immediately to the following result.

Theorem 2.6. *$\mathbb{S}q(\mathbb{C})$ is connected with diameter 2. In fact, $\overline{\mathbb{S}q(\mathbb{C})}$ is a disjoint union of infinitely many copies of K_2.*

Remark 2.7. Note that we can readily see $\mathbb{S}q(\mathbb{R}) \ncong \mathbb{S}q(\mathbb{C})$, as every vertex in the complement of $\mathbb{S}q(\mathbb{C})$ has degree 1, whereas every vertex in the complement of $\mathbb{S}q(\mathbb{R})$ is of infinite degree.

It is a well-known fact that every finite connected graph has a spanning tree. The same is not necessarily true for an infinite graph. We show in our next result that if F is a field such that $|F|$ is countably infinite, then $\mathbb{S}q(F)$ has a spanning tree.

Proposition 2.8. *Let F be field such that $|F| = \aleph_0$. Then $\mathbb{S}q(F)$ has a rooted spanning tree T. Also, the root of T is of infinite degree, while the other vertices of T have degree $1, 2, 3$ or 4.*

Proof. Clearly, $Char(F) = 0$ and hence $\mathbb{Q} \subseteq F$. Consider the vertex -1. Let T_0 be the subgraph induced by the edges joining u and -1 for all neighbors u of -1. Let $F - (\{0, -2\} \cup N(-1))$ be enumerated as $\{b_n \mid n \in \mathbb{N}\}$. We have a path $b_n \leftrightarrow 4^{-1}b_n^2 + 1 \leftrightarrow -1$ for each $n \in \mathbb{N}$. So $4^{-1}b_n^2 + 1$ is one of the neighbors of -1. For each $n \in \mathbb{N}$, let T_n be the subgraph induced by T_0 and the edge joining b_n and $4^{-1}b_n^2 + 1$. Then $T' = \bigcup_{n\in\mathbb{N}} T_n$ contains all the vertices of $\mathbb{S}q(F)$ except -2. Let $T = T' \cup \{e\}$, where e is the edge joining -2 and 3. Clearly, T is a tree (rooted at -1) which contains all vertices of $\mathbb{S}q(F)$, i.e., T is a rooted spanning tree. The vertex -1 has infinitely many neighbors and each neighbor ($\neq 3$) has degree 2 or 3 (as each neighbor of -1 can be of the form $4^{-1}b_n^2 + 1$ for at most two values of n, keeping in mind that $b_n^2 = (-b_n)^2$). If 3 is a neighbor of -1, then $deg(3) = 4$ if 8 is a square element. In this regard it is easy to see that for any vertex $v (\neq -1)$ we have a path of length at most 3 between -1 and v in T. So we have found a rooted spanning tree where the root is of infinite degree and the other vertices are of degree 1, 2, 3 or 4. ∎

Next, we consider the rings $\mathbb{Z}[\sqrt{d}]$ where d is not a perfect square.

Proposition 2.9. *Let $d \in \mathbb{N} - \{m^2 \mid m \in \mathbb{N}\}$. Then $\mathbb{S}q(\mathbb{Z}[\sqrt{d}])$ is not connected* *and it has exactly two components.*

Proof. Let $A = \{a + 2k\sqrt{d} \mid a, k \in \mathbb{Z}\}$. Clearly, $\mathbb{Z} \subset A$. Now for $a \neq -1$, we have a path $a + 2k\sqrt{d} \leftrightarrow k^2 d - 2k\sqrt{d} - a \leftrightarrow 1 + a$. Again, we have a path $-1+2k\sqrt{d} \leftrightarrow d-2k\sqrt{d}+1 \leftrightarrow k^2-1+4k\sqrt{d} \leftrightarrow 4d+1$. So for each vertex v in A, there is path from v to some integer. The connectedness of $\mathbb{S}q(\mathbb{Z})$ then implies that the vertices belonging to the set A are in the same component in $\mathbb{S}q(\mathbb{Z}[\sqrt{d}])$. Again, let $B = \{a+(2k+1)\sqrt{d} \mid a, k \in \mathbb{Z}\}$. If $a = 4m+2$ for some $m \in \mathbb{Z}$, then we have a path $a+\sqrt{d} \leftrightarrow 4m^2-1-\sqrt{d} \leftrightarrow 1+\sqrt{d}$. We note that for any $a+(2k+1)\sqrt{d}$, there exists a path $a+(2k+1)\sqrt{d} \leftrightarrow k^2d-\sqrt{d}+1-a \leftrightarrow a-1+\sqrt{d} \leftrightarrow 2-a-\sqrt{d} \leftrightarrow 2 + a + \sqrt{d} \leftrightarrow 7 - a - \sqrt{d} \leftrightarrow -3 + a + \sqrt{d} \leftrightarrow 4 - a - \sqrt{d} \leftrightarrow a + \sqrt{d}$. Now one of $a + \sqrt{d}, a - 1 + \sqrt{d}, a - 3 + \sqrt{d}$ and $a + 2 + \sqrt{d}$ must be of the form $4m + 2 + \sqrt{d}$. This shows that we have a path from $a+(2k+1)\sqrt{d}$ to $1+\sqrt{d}$. Thus the vertices of B are in a single component of $\mathbb{S}q(\mathbb{Z}[\sqrt{d}])$. Finally, we show that there is no path between the vertices $1+2\sqrt{d}$ and $1+3\sqrt{d}$ (clearly, they are not adjacent). If possible, let $1+2\sqrt{d} \leftrightarrow c_1 \leftrightarrow c_2 \leftrightarrow \cdots \leftrightarrow c_k \leftrightarrow 1+3\sqrt{d}$ be a path between $1+2\sqrt{d}$ and $1 + 3\sqrt{d}$. So we must have $1 + 2\sqrt{d} + c_1 = h_1^2, c_1 + c_2 = h_2^2, \ldots, c_{k-1} + c_k = h_k^2, c_k + 1 + 3\sqrt{d} = h_{k+1}^2$ for some $h_1, h_2, \ldots, h_k \in \mathbb{Z}[\sqrt{d}] - \{0\}$. This implies that $(1 + 2\sqrt{d}) + (-1)^{k+1}(1 + 3\sqrt{d}) = h_1^2 - h_2^2 + \cdots + (-1)^{k+1}h_{k+1}^2$. Here the coefficient of $\sqrt{d}$ in the left hand side is odd whereas in the right hand side, it is even. Thus, the vertices $1+2\sqrt{d}$ and $1+3\sqrt{d}$ are in different components. Hence $\mathbb{S}q(\mathbb{Z}[\sqrt{d}])$ is not connected and it has exactly two components (corresponding to vertices belonging to A and B respectively). ∎

The analogous result for $\mathbb{Z}[\sqrt{-d}]$ (where d is not a perfect square) also holds.

Proposition 2.10. *Let $d \in \mathbb{N} - \{m^2 \mid m \in \mathbb{N}\}$. Then $\mathbb{S}q(\mathbb{Z}[\sqrt{-d}])$ is not connected and it has exactly two components.*

Interestingly, the same result is true for $\mathbb{Z}[i]$ as well, as shown next.

Proposition 2.11. $\mathbb{S}q(\mathbb{Z}[i])$ *is not connected and it has exactly 2 components.*

Proof. It is easy to see that if $\mathbb{S}q(\mathbb{Z}[i])$ is connected, then we must have some $a_1, a_2, k, m \in \mathbb{Z}$ such that $a_1 + 2ki$ and $a_2 + (2m + 1)i$ are adjacent in $\mathbb{S}q(\mathbb{Z}[i])$. In that case, $a_1 + a_2 + (2k + 2m + 1)i = (c + di)^2$ for some $c, d \in \mathbb{Z}$. This implies that $2k + 2m + 1 = 2cd$ (equating the coefficients of i in both sides), which is a contradiction as the left hand side is an odd integer whereas the right hand side gives an even integer. So $\mathbb{S}q(\mathbb{Z}[i])$ is not connected. Proceeding similarly to what we did in Proposition 2.9, one can show that $\mathbb{S}q(\mathbb{Z}[i])$ has exactly two components corresponding to the sets of vertices $\{a + 2ki \mid a, k \in \mathbb{Z}\}$ and $\{a + (2k + 1)i \mid a, k \in \mathbb{Z}\}$, respectively. ∎

Remark 2.12. We have seen in [12] that there might exist a chain of subrings

$R_1 \subset R_2 \subset R_3$ such that $\mathbb{S}q(R_1)$ and $\mathbb{S}q(R_3)$ are connected, but $\mathbb{S}q(R_2)$ is not connected. For example, $\mathbb{Z} \subset \mathbb{Z}[x] \subset Q(\mathbb{Z}[x])$ and we have that $\mathbb{S}q(\mathbb{Z})$ and $\mathbb{S}q(Q(\mathbb{Z}[x]))$ are both connected whereas $\mathbb{S}q(\mathbb{Z}[x])$ is not. Now we know that $\mathbb{Z} \subset \mathbb{Z}[\sqrt{d}] \subset \mathbb{R}$ as subrings (where d is any non-square natural number). We have shown that $\mathbb{S}q(\mathbb{Z})$ and $\mathbb{S}q(\mathbb{R})$ are connected (cf. Theorems 2.1 and 2.5), but $\mathbb{S}q(\mathbb{Z}[\sqrt{d}])$ is not connected (cf. Proposition 2.9). Again, $\mathbb{Z} \subset \mathbb{Z}[i] \subset \mathbb{C}$ as subrings. We have shown that $\mathbb{S}q(\mathbb{Z})$ and $\mathbb{S}q(\mathbb{C})$ are connected (cf. Theorem 2.6), whereas $\mathbb{S}q(\mathbb{Z}[i])$ is not connected (cf. Proposition 2.11). Thus we have found two more examples showing that we might have chain of subrings $R_1 \subset R_2 \subset R_3$ such that $\mathbb{S}q(R_1)$ and $\mathbb{S}q(R_3)$ are connected but $\mathbb{S}q(R_2)$ is not connected.

Interestingly, the technique used in Theorem 2.2 can be used to prove the connectedness of $\mathbb{S}q(R)$ for some rings R with $Char(R) = 0$ which are not fields. The following is one such instance.

Theorem 2.13. $\mathbb{S}q(C[0,1])$ *is connected with diameter at most 7.*

Proof. For a function $f \in C[0,1]$ and any $t \in \mathbb{N}$, let $\frac{1}{t}f$ denote the function g, where $g(x) = \frac{1}{t}f(x)$. Clearly, $\frac{1}{t}f \in C[0,1]$ for all $t \in \mathbb{N}$ and $f \in C[0,1]$. For every $r \in \mathbb{R}$, let f_r denote the function given by $f_r(x) = r$ for all $x \in [0,1]$. Clearly, f_0 is the zero-element of $C[0,1]$, and $f_{-r} = -f_r$ for all $r \in \mathbb{R}$. Now proceeding exactly in the same way as we did in Theorem 2.2, we can show that for every $f \in C[0,1] - \{f_0, f_{-1}\}$, there is a path between f and f_{-1} in $\mathbb{S}q(C[0,1])$. Thus $\mathbb{S}q(C[0,1])$ is connected and arguing similarly as in Theorem 2.2, we see that $diam(\mathbb{S}q(C[0,1])) \leq 7$. ∎

It is interesting to consider the connectedness of the square element graph over a ring in relation with the connectedness of the square element graphs taken over its quotient rings. We have seen that $\mathbb{S}q(\mathbb{Z})$ is connected. We shall shortly show that there are ideals I of $\mathbb{Z}$ for which $\mathbb{S}q(\mathbb{Z}/I)$ is not connected. Before that, we have the following result.

Theorem 2.14. $\mathbb{S}q(k\mathbb{Z})$ *is not connected for any natural number* $k > 1$.

Proof. We show that there is no path between the vertices k and $-k^2$ in $\mathbb{S}q(k\mathbb{Z})$. As $-k^2 + k < 0$, $k \not\leftrightarrow -k^2$. If possible, let there be a path $k \leftrightarrow c_1 \leftrightarrow c_2 \leftrightarrow \cdots \leftrightarrow c_m \leftrightarrow -k^2$. This implies that $k + c_1 = (kh_1)^2, c_1 + c_2 = (kh_2)^2, \ldots, c_{m-1} + c_m = (kh_m)^2, c_m + (-k^2) = (kh_{m+1})^2$ for some $h_1, h_2, \ldots, h_{m+1} \in \mathbb{Z} - \{0\}$. Now $k + (-1)^{m+1}(-k^2) = k + c_1 - (c_1 + c_2) + (c_2 + c_3) - \ldots + (-1)^{m+1}(c_m - k^2) = (kh_1)^2 - (kh_2)^2 + \ldots + (-1)^{m+1}(kh_{m+1})^2$. Here the right hand side is a multiple of k^2 but the left hand side is not, which is a contradiction. Thus there is no path between k and $-k^2$. So $\mathbb{S}q(k\mathbb{Z})$ is not connected. ∎

Remark 2.15. (i) We note that $\mathbb{S}q(\mathbb{Z}_{10})$ is connected (cf. [11]). Consider the ideal $I = \{\bar{0}, \bar{5}\}$ of $\mathbb{Z}_{10}$. The graph $\mathbb{S}q(I)$ is also connected. However, $\mathbb{S}q(\mathbb{Z}_{10}/I)(\cong \mathbb{S}q(\mathbb{Z}_5))$ is not connected (cf. [12]). This shows that for a ring R and an ideal

I of R, connectedness of $\mathbb{S}q(R)$ and $\mathbb{S}q(I)$ does not imply the connectedness of $\mathbb{S}q(R/I)$.

(ii) We have shown that $\mathbb{S}q(\mathbb{Z})$ is connected. Considering the ideal $I = 6\mathbb{Z}$, we note that the graph $\mathbb{S}q(\mathbb{Z}/I)(\cong \mathbb{S}q(\mathbb{Z}_6))$ is connected. But as we know from Theorem 2.14, the graph $\mathbb{S}q(6\mathbb{Z})$ is not connected. So connectedness of $\mathbb{S}q(R)$ and $\mathbb{S}q(R/I)$ does not imply the connectedness of $\mathbb{S}q(I)$.

The above remark leads us naturally to the question if the connectedness of $\mathbb{S}q(I)$ and $\mathbb{S}q(R/I)$ implies the connectedness of $\mathbb{S}q(R)$. As we show in the next theorem, it does hold true when R is commutative and unital.

Theorem 2.16. *Let R be a commutative ring with 1 and let I be an ideal of R. If $\mathbb{S}q(I)$ and $\mathbb{S}q(R/I)$ are connected, then $\mathbb{S}q(R)$ is connected.*

Proof. Let $a \in R-I$. We look for a path from the vertex a to i_k for some $i_k \in I$. If $a + I = b^2 + I$ for some $b \in R$, then $a + i_k = b^2$ for some $i_k \in I$ and hence $a \leftrightarrow i_k$. If there is no $b \in I$ for which $a + I = b^2 + I$, then the connectedness of $\mathbb{S}q(R/I)$ implies that there is a path between $a+I$ and $1+I$ (note that $a \neq 1$, as $1+I = 1^2+I$). Let that path be $a+I \leftrightarrow c_1+I \leftrightarrow \cdots \leftrightarrow c_k+I \leftrightarrow b^2+I$. Then $a+c_1+I = h_1^2+I, c_2+c_1+I = h_2^2+I, \ldots, c_{k-1}+c_k+I = h_k^2+I, c_k+b^2+I = h_{k+1}^2+I$ for some $h_1, h_2, \ldots, h_k \in R$. So we find that $a+(c_1-i_1) = h_1^2, c_1-i_1+(c_2-i_2) = h_2^2, \ldots, c_k-i_k+(b^2-i_{k+1}) = h_{k+1}^2$ for some $i_1, i_2, \ldots, i_{k+1} \in I$. Thus we have a path $a \leftrightarrow c_1-i_1 \leftrightarrow c_2-i_2 \leftrightarrow \cdots \leftrightarrow c_k-i_k \leftrightarrow b^2-i_{k+1} \leftrightarrow i_{k+1}$. So there is always a path from a to a vertex belonging to I. Now as the vertices in I are in the same component (since $\mathbb{S}q(I)$ is connected), this implies that $\mathbb{S}q(R)$ is connected. ∎

3. $\mathbb{S}q(R)$ over Polynomial Rings and Matrix Rings

In this section we consider the square element graph over some special polynomial rings and matrix rings.

We have seen in [12] that for a field F with $Char(F) = 2$, $\mathbb{S}q(F)$ is complete if F is finite; and $\mathbb{S}q(F)$ is either complete or disconnected if F is infinite. However, when we consider the polynomial ring $F[x]$ over any field F of characteristic 2, the graph $\mathbb{S}q(F[x])$ is disconnected, as we show next.

Proposition 3.1. *Let F be a field with $Char(F) = 2$. Then $\mathbb{S}q(F[x])$ is disconnected.*

Proof. An element $a_0 + a_1x^{r_1} + a_2x^{r_2} + \cdots + a_tx^{r_t}$ (where $t \in \mathbb{N}$) is a square element in $F[x]$ if and only if the r_i's are even for all $i = 1, 2, \ldots, t$. This shows that a sum of a square and a non-square cannot be square. Now for connectedness of $\mathbb{S}q(F[x])$, it is necessary to have at least one edge between a square vertex and a non-square vertex. Hence $\mathbb{S}q(F[x])$ is not connected. ∎

Note that as a corollary of the above result, it follows that $\mathbb{S}q(\mathbb{Z}_2[x])$ is disconnected, an alternative proof of which was given by Sen Gupta and Sen in [12]. Now when we move to a field F with characteristic not equal to 2, $\mathbb{S}q(F[x])$ is connected.

Proposition 3.2. *If F is a field with $Char(F) \neq 2$, then $\mathbb{S}q(F[x])$ is connected.*

Proof. First, let $Char(F) = p$ for some odd prime p. If $f \in F[x] - \{-1\}$, then $(1 + \frac{p+1}{2}f)^2 = f + 1 + \frac{(p+1)^2}{4}f^2$. Now $1 + \frac{p+1}{2}f = 0$ implies that $f = -2$, and $1 + \frac{(p+1)^2}{4}f^2 = 0$ implies that $f^2 = -4$. Let $f \in F[x] - \{0\}$ be such that $f \neq -2$ and $f^2 \neq -4$. Then we have a path $f \leftrightarrow 1 + \frac{(p+1)^2}{4}f^2 \leftrightarrow -1$. Again, if $g = -2$ or $g^2 = -4$, then we have a path $g \leftrightarrow -g + x^2 \leftrightarrow 1 + \frac{(p+1)^2}{4}(-g + x^2)^2 \leftrightarrow -1$. So for every $f \in F[x] - \{1\}$, there is a path between f and -1 in $\mathbb{S}q(F[x])$. Consequently, $\mathbb{S}q(F[x])$ is connected. Next, let $Char(F) = 0$. Let $f \in F[x] - \{0, -1, -2\}$. Then we have a path $f \leftrightarrow 1 + \frac{f^2}{4} \leftrightarrow -1$. Again, if $g = -2$, we have a path $g \leftrightarrow 3 \leftrightarrow 1 + \frac{9}{4} \leftrightarrow -1$. Thus every vertex of $\mathbb{S}q(F[x])$ is in the same component with -1. So $\mathbb{S}q(F[x])$ is connected. ∎

Combining the above two results, we have the following result:

Theorem 3.3. *For a field F, $\mathbb{S}q(F[x])$ is connected if and only if $Char(F) \neq 2$.*

Next, we show that $\mathbb{S}q(F[x])$ always contains a 3-cycle.

Theorem 3.4. *$girth(\mathbb{S}q(F[x]) = 3$ for any field F.*

Proof. If $Char(F) = 0$, then $-1 \leftrightarrow 1 + 4^{-1}x^2 \leftrightarrow 1 + 4^{-1}(1 + 4^{-1}x^2)^2 \leftrightarrow -1$ is a 3-cycle. Next, let $Char(F) = 2$. Then $x^2 \leftrightarrow x^4 \leftrightarrow x^6 \leftrightarrow x^2$ is a 3-cycle. Finally, if $Char(F) = p$ (where p is an odd prime), then we have a 3-cycle $-1 \leftrightarrow 1 + (2^{-1}(p + 1))^2x^2 \leftrightarrow 1 + (1 + (2^{-1}(p + 1))^2x^2)^2 \leftrightarrow -1$. Hence $girth(\mathbb{S}q(F[x]) = 3$. ∎

Now we consider $\mathbb{S}q(R)$ over matrix rings. In [12], Sen Gupta and Sen had shown that $\mathbb{S}q(M_2(\mathbb{Z}_2))$ is connected. We now show that $\mathbb{S}q(M_n(F))$ is in fact connected for any finite field F with characteristic 2. For this, we first have the following lemma.

Lemma 3.5. *If F is a finite field with $Char(F) = 2$, then every non-square element of $M_n(F)$ (where $n > 1$) can be expressed as a sum of squares.*

Proof. As F is a finite field with $Char(F) = 2$, we find that $|F| = 2^r$ (for some $r \in \mathbb{N}$). If $(F - \{0\}, \cdot) =< a >$, then $a^{2r-1} = 1$. This shows that every non-zero element of F can be expressed as an even power of a. Consequently, each non-zero element of F is a square element. We now use the principle of Mathematical Induction to show that every non-square element of $M_n(F)$ can be expressed as a sum of squares. First, let $n = 2$. Due to the above reasoning, any matrix in $M_n(F)$ can be expressed as $\begin{bmatrix} x^2 & y^2 \\ z^2 & t^2 \end{bmatrix}$, which can again be expressed as $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}^2 + \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}^2 + \begin{bmatrix} 0 & y \\ 0 & y \end{bmatrix}^2 + \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}^2 + \begin{bmatrix} 0 & 0 \\ z & z \end{bmatrix}^2 + \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}^2$. So the hypothesis is true for $n = 2$. Now let the hypothesis be true for $n = m$ for some $m > 2$. Consider a non-square matrix A in $M_{m+1}(F)$. Then we have

$$
\begin{aligned}
A &= \left[\begin{array}{cccc|c}
a_{11}^2 & a_{12}^2 & \cdots & a_{1m}^2 & a_{1m+1}^2 \\
a_{21}^2 & a_{22}^2 & \cdots & a_{2m}^2 & a_{2m+1}^2 \\
\cdots & & \cdots & \cdots\cdots & \\
a_{m1}^2 & a_{m2}^2 & \cdots & a_{mm}^2 & a_{mm+1}^2 \\
\hline
a_{m+11}^2 & a_{m+12}^2 & \cdots & a_{m+1m}^2 & a_{m+1m+1}^2
\end{array}\right] \qquad (\text{where } \ a_{ij} \in F) \\
&= \left[\begin{array}{cccc|c}
a_{11}^2 & a_{12}^2 & \cdots & a_{1m}^2 & 0 \\
a_{21}^2 & a_{22}^2 & \cdots & a_{2m}^2 & 0 \\
\vdots & & \ddots & \vdots * \vdots & \\
a_{m1}^2 & a_{m2}^2 & \cdots & a_{mm}^2 & 0 \\
\hline
0 & 0 & \cdots & 0 & 0
\end{array}\right] + \left[\begin{array}{cccc|c}
0 & 0 & \cdots & 0 & a_{1m+1}^2 \\
0 & 0 & \cdots & 0 & a_{2m+1}^2 \\
\cdots & & \cdots & \cdots\cdots & \\
0 & 0 & \cdots & 0 & a_{mm+1}^2 \\
\hline
a_{m+11}^2 & a_{m+12}^2 & \cdots & a_{m+1m}^2 & a_{m+1m+1}^2
\end{array}\right] \\
&= I + J
\end{aligned}
$$

Now let $H = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & \cdots & a_{1m}^2 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & \cdots & a_{2m}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}^2 & a_{m2}^2 & a_{m3}^2 & \cdots & a_{mm}^2 \end{bmatrix}$. Then from the induction hypothesis $H = H_1^2 + H_2^2 + \ldots + H_k^2$ where $k \geq 1$, and $H_i \in M_m(F)$.

So $I = \left[\begin{array}{c|c} H & O_{m1} \\ \hline O_{1m} & O_{11} \end{array}\right] = \left[\begin{array}{c|c} H_1^2 + H_2^2 + \ldots. + H_k^2 & O_{m1} \\ \hline O_{1m} & O_{11} \end{array}\right] = T_1^2 + T_2^2 + \cdots + T_k^2$, where $T_i = \left[\begin{array}{c|c} H_i & O_{m1} \\ \hline O_{1m} & O_{11} \end{array}\right]$ for $i = 1, 2, \ldots, k$.

On the other hand, we find that

$$
\begin{aligned}
J &= \left[\begin{array}{cccc|c}
0 & 0 & \cdots & 0 & a_{1m+1}^2 \\
0 & 0 & \cdots & 0 & a_{2m+1}^2 \\
\cdots & & \cdots & \cdots\cdots & \\
0 & 0 & \cdots & 0 & a_{mm+1}^2 \\
\hline
a_{m+11}^2 & a_{m+12}^2 & \cdots & a_{m+1m}^2 & a_{m+1m+1}^2
\end{array}\right] \\
&= \left[\begin{array}{cccc|c}
0 & 0 & \cdots & 0 & a_{1m+1}^2 \\
0 & 0 & \cdots & 0 & a_{2m+1}^2 \\
\cdots & & \cdots & \cdots\cdots & \\
0 & 0 & \cdots & 0 & a_{mm+1}^2 \\
\hline
0 & 0 & \cdots & 0 & a_{m+1,m+1}^2
\end{array}\right] + \left[\begin{array}{cccc|c}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\cdots & & \cdots & \cdots\cdots & \\
0 & 0 & \cdots & 0 & 0 \\
\hline
a_{m+11}^2 & a_{m+12}^2 & \cdots & a_{m+1m}^2 & 0
\end{array}\right]
\end{aligned}
$$

$$= \left[\begin{array}{ccc|c} 0 & \cdots & 0 & a_{1m+1}^2 \\ 0 & \cdots & 0 & a_{2m+1}^2 \\ \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & a_{mm+1}^2 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]^2 + \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & 0 \\ \hline a_{m+11}^2 & \cdots & a_{m+1m}^2 & 1 \end{array}\right]^2 + \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & a_{m+1m+1} \end{array}\right]^2$$

$$= S_1^2 + S_2^2.$$

So $A = I + J = T_1^2 + T_2^2 + \cdots + T_k^2 + S_1^2 + S_2^2 + S_3^2$. This shows that the hypothesis is true for $n = m+1$ as well. Hence the result holds by the principle of Mathematical Induction. ∎

Theorem 3.6. *Let F be a finite field with $Char(F) = 2$. Then $\mathbb{S}q(M_n(F))$ is connected.*

Proof. We show that for any non-zero matrix $A \in M_n(F)$, there is a path between A and I_n. If $A = B^2$ for some non-zero $B \in M_n(F)$, then $A \leftrightarrow I_n$ since $A+I_n = B^2+I_n = (B+I_n)^2$. So every square vertex is adjacent to I_n. Again, if A is not a square element in $M_n(F)$, then A can be expressed as $S_1^2+S_2^2+\ldots+S_k^2$ (by Lemma 3.5), where $S_i \in M_n(F)$ for $i = 1, 2, \ldots, k$. Then we have a path between A and I_n given by $A \leftrightarrow S_2^2+S_3^2+\ldots+S_k^2 \leftrightarrow S_3^2+S_4^2+\ldots+S_k^2 \leftrightarrow \cdots \leftrightarrow S_k^2 \leftrightarrow I_n$. Thus $\mathbb{S}q(M_n(F))$ is connected. ∎

In fact, $\mathbb{S}q(M_n(F))$ is connected for any field F with odd characteristic also.

Theorem 3.7. *Let p be an odd prime and $n \in \mathbb{N} - \{1\}$. If F is a field with $Char(F) = p$, then $\mathbb{S}q(M_n(F))$ is connected.*

Proof. Let I be the identity and 0_n be the zero-element of $M_n(F)$. We show that there exists a path from $-(2^{-1}(p+1))^2 I$ to any other vertex A of $\mathbb{S}q(M_n(F))$.

Case 1: Let $(2^{-1}(p+1))^2 I + A^2, (2^{-1}(p+1))I + A \neq 0_n$. Then we have a path $A \leftrightarrow A^2 + (2^{-1}(p+1))^2 I \leftrightarrow -(2^{-1}(p+1))^2 I$.

Case 2: Let $(2^{-1}(p+1))^2 I + A^2 = 0_n$ and $(2^{-1}(p+1))I + A \neq 0_n$. If $(I - A)^2 + (2^{-1}(p+1))^2 I = 0_n = (2^{-1}(p+1))^2 I + A^2$, then we have that $A = 2^{-1} I$, which is a contradiction since $(2^{-1} I)^2 + (2^{-1}(p+1))I \neq 0_n$. Again if $(I - A) + (2^{-1}(p+1))I = 0_n$, then we have that $A = (2^{-1}(p+3))I$, which is again a contradiction since $((2^{-1}(p+3))I)^2 + (2^{-1}(p+1))^2 I \neq 0_n$. Also, $I \neq A$. So we have a path $A \leftrightarrow I - A \leftrightarrow (I - A)^2 + (2^{-1}(p+1))^2 I \leftrightarrow -(2^{-1}(p+1))^2 I$.

Case 3: Let $(2^{-1}(p+1))^2 I + A^2 \neq 0_n$ and $(2^{-1}(p+1))I + A = 0_n$ (note that both cannot be zero together). Then $A = -(2^{-1}(p+1))I$. So we have a path $-(2^{-1}(p+1))I \leftrightarrow I + (2^{-1}(p+1))I \leftrightarrow (I + (2^{-1}(p+1))I)^2 + (2^{-1}(p+1))^2 I \leftrightarrow -(2^{-1}(p+1))^2 I$.

So there is path from $-2^{-1}(p+1)I$ to all vertices of $\mathbb{S}q(M_n(F))$, and thus $\mathbb{S}q(M_n(F))$ is connected. ∎

From Theorems 3.6 and 3.7, we have the following result:

Corollary 3.8. $\mathbb{S}q(M_n(F))$ *is connected for any finite field* F*, where* $n > 1$*.*

Finally, we show that $\mathbb{S}q(M_n(F))$ is connected for any field F with characteristic 0 as well.

Theorem 3.9. *If* F *is a field with* $Char(F) = 0$*, then* $\mathbb{S}q(M_n(F))$ *is connected for any* $n \in \mathbb{N} - \{1\}$*.*

Proof. Let I be the identity and 0_n be the zero-element of $M_n(F)$. Now we show that all vertices are in the same component with the vertex $-4^{-1}I$ in $\mathbb{S}q(M_n(F))$.

Case 1: If $4^{-1}I + A^2 \neq 0_n$ and $2^{-1}I + A \neq 0$, then we have a path $A \leftrightarrow A^2 + 4^{-1}I \leftrightarrow -4^{-1}I$.

Case 2: Let $4^{-1}I + A^2 = 0$ and $2^{-1}I + A \neq 0$. If $(I - A)^2 + 4^{-1}I = 0 = 4^{-1}I + A^2$, then we have that $A = 2^{-1}I$ which is a contradiction since $(2^{-1}I)^2 + 4^{-1}I \neq 0$. Again, if $(I - A) + 2^{-1}I = 0$ then this implies that $A = 2^{-1}3I$, which is again a contradiction since $(2^{-1}3I)^2 + 4^{-1}I \neq 0$. So we do have a path $A \leftrightarrow I - A \leftrightarrow (I - A)^2 + 4^{-1}I \leftrightarrow -4^{-1}I$.

Case 3: Let $4^{-1}I + A^2 \neq 0$ and $2^{-1}I + A = 0$ (note that both cannot be zero together). Then $A = -2^{-1}I$. In this case we have a path $-2^{-1}I \leftrightarrow 4^{-1}3I \leftrightarrow 4^{-1}I + (4^{-1}3I)^2 \leftrightarrow -4^{-1}I$.

The above three cases show that from any vertex we can always have a path to the vertex $-4^{-1}I$ in $\mathbb{S}q(M_n(F))$. So $\mathbb{S}q(M_n(F))$ is connected. ∎

Combining Theorems 3.7 and 3.9, we obtain the following result:

Proposition 3.10. *If* $n \in \mathbb{N} - \{1\}$ *and* F *is a field with* $Char(F) \neq 2$*, then* $\mathbb{S}q(M_n(F))$ *is connected.*

We conclude the paper with the following question:

Problem 3.11. Characterize the class of rings R for which $\mathbb{S}q(M_n(R))$ is connected.

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